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SEMILINEAR PARABOLIC EQUATIONS WITHOUT INERTIAL MANIFOLD

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OUTLINE OF THE TALK

- ▲ Semilinear parabolic equations
- ▲ Inertial manifolds of parabolic equations
- ▲ Sufficient conditions for the existence of an inertial manifold
- ▲ Necessary condition for the existence of an inertial manifold
- ▲ Equation with nonlocal diffusion without an inertial manifold
- ▲ Normally hyperbolic inertial manifolds
- ▲ Reaction–diffusion system without a normally hyperbolic inertial manifold
- ▲ Inertial manifold without normal hyperbolicity
- ▲ Absolutely normally hyperbolic inertial manifolds

OBJECT OF STUDY

In a real separable Hilbert space $(X, \|\cdot\|)$, $\dim X = \infty$, consider the **semilinear parabolic equation**

$$u_t = -Au + F(u) \qquad (*)$$

Here

1. $A: D(A) \to X$ is a linear self-adjoint positive operator with compact inverse A^{-1}.
2. $F: H \to X$ *is a smooth nonlinear function with domain* $H = D(A^{\theta})$, $0 \le \theta < 1$, $\| u \|_H = \left\| A^{\theta} u \right\|$,

$$\| F(u) - F(v) \| \le K(r) \| u - v \|_H \text{ for } \| u \|_H \le r, \ \| v \|_H \le r.$$

3. *There exists a smooth dissipative phase semiflow* $\Phi_t : H \to H$.

The dissipativity means the existence of an **absorbing ball** in the phase space H. Since the **nonlinearity exponent** $\theta < 1$, we see that the function F is “weaker” than the operator A, which means that Eq. (*) is **semilinear**. We have $H = X$ if $\theta = 0$.

Nonlinearities $F: H \to X$ with the properties described above will be called **admissible** nonlinearities. The **compact attractor** $\mathcal{A} \subset H$ is the collection of all complete bounded trajectories.

INERTIAL MANIFOLDS
OF PARABOLIC EQUATIONS

The **inertial manifold** of the semilinear parabolic equation $(*)$ is a smooth **finite-dimensional** invariant surface $M \subset H$ that contains the global attractor and attracts all trajectories at large time **with exponential tracking**. Usually, M have globally Cartesian structure and M is diffeomorphic to $\mathbb{R}^n$. The restriction of the parabolic equation to M is an **ordinary differential equation (inertial form) in $\mathbb{R}^n$ which completely describes the eventual dynamics of the system**.

The existence of an inertial manifold implies that the eventual behavior of an infinite-dimensional dynamical system is controlled by **finitely many** parameters.

CONCLUSION: a system with infinitely many degrees of freedom essentially has finitely many degrees of freedom as $t \to +\infty$.

HISTORY OF THE TOPIC

The term "inertial manifold" was introduced in the note [1]. Essentially, this object had already been considered in [2,3]. Mane's paper [4] is apparently the first study on the topic. The contemporary state of the topic: [5].

PARADOX: Nothing is known about inertial manifolds for a majority of equations of mathematical physics.

Namely, it has been possible to establish the existence of inertial manifolds for a narrow class of parabolic equations, while known examples [6,7] in which there is no inertial manifold seem to be artificial and are not related to practically important problems.

[1] C. Foias, G.R. Sell, and R. Temam. *C. R. Acad. Sci. Paris I*, **301**:5 (1985), 139–141.

[2] D. Henry. *Geometric theory of semilinear parabolic equations*, Lecture Notes in Math., **840**, 1981.

[3] X. Mora. Contemp. Math., **17** (1983), 353–360.

[4] R. Mane. Lecture Notes in Math., **597**, 1977, 361–378.

[5] S. Zelik. *Proc. Roy. Soc. Edinburgh, Ser.* A, **144**:6 (2014), 1245–1327.

[6] A.V. Romanov. *Math. Notes*, **68**:3–4 (2000), 378–385.

[7] A. Eden, V. Kalantarov, and S. Zelik. *Russian Math. Surveys*, **68**:2 (2013), 99–226.

The main goal of the study

is to construct examples of parabolic equations of

mathematical physics that do not have

an inertial manifold

INERTIAL MANIFOLDS: SUFFICIENCY

The only *general* sufficient condition for the existence of an inertial manifold $M \subset H$ of the equation $u_t = -Au + F(u)$ **for an arbitrary admissible nonlinearity** F is **the spectrum sparseness condition** for the linear part of the equation (e.g., see [1]):

$$\sup_{n\geq 1} \frac{\mu_{n+1}-\mu_n}{\mu_{n+1}^{\theta}+\mu_n^{\theta}} = \infty, \text{ where } \{0<\mu_1 \leq \mu_2 < \ldots\} = \sigma(A).$$

For the **reaction–diffusion equation**

$$\partial_t u = \Delta u + f(x,u)$$

in a bounded domain $\Omega \subset \mathbb{R}^m$, one has $A = -\Delta$, $\theta = 0$, $\mu_n \sim cn^{2/m}$, so that the spectrum sparseness condition $\sup\limits_{n\geq 1}(\mu_{n+1}-\mu_n) = \infty$ holds only in one-dimensional and (rarely) two-dimensional problems.

For the Beltrami–Laplace operator on the sphere S^m **the spectrum sparseness condition** $\sup\limits_{n\geq 1}(\mu_{n+1}-\mu_n) = \infty$ **holds** $\forall m \geq 2$ **!**

[1] S. Zelik. *Proc. Roy. Soc. Edinburgh, Ser. A*, **144**:6 (2014), 1245-1327.

HOW CAN WE AVOID THE SPECTRUM SPARSENESS CONDITION ?

The **spatial averaging principle** for the Laplacian Δ in bounded domain $\Omega \subset \mathbb{R}^m$ $(m \le 3)$ suggested in [1] sometimes permits one to construct inertial manifolds **avoiding the spectrum sparseness condition**. It is the following property: for $\forall h \in H^2(\Omega)$ operator $\Delta + h(x)$ can be well approximated by $\Delta + \bar{h}$ over "large segments" of $L^2(\Omega)$, where $\bar{h} = (\mathrm{vol}\,\Omega)^{-1} \int_\Omega h(x)dx$. **This property follows from the spectrum sparseness condition.** The corresponding method was used in [1] to prove the existence of an inertial manifold for the scalar reaction–diffusion equation

$$\partial_t u = \Delta u + f(x,u), \qquad f \in C^3,$$

in cube $\Omega = (0, 2\pi)^3$ and in rectangle $\Omega = (0,a) \times (0,b)$ with boundary conditions (N), (D) or (P). Analogical results were obtained [2] for some bounded domains $\Omega \subset \mathbb{R}^m$ $(m = 2,3)$. The abstract scheme of this method was suggested in [3] and successfully applied in [4] to the Cahn–Hilliard equation

$$\partial_t u = -\Delta(\Delta u - f(u)), \quad f \in C^3,$$

on the 3D torus.

[1] J. Mallet-Paret and G.R. Sell. *J. Amer. Math. Soc.*, **1**:4 (1988), 805–866.

[2] H. Kwean. *Int. J. Math. Math. Sci.*, **28**:5 (2001), 293–299.

[3] S. Zelik. *Proc. Roy. Soc. Edinburgh*, *Ser.* A, **144**:6 (2014), 1245–1327.

[4] A. Kostianko and S. Zelik. *Comm. Pure Appl. Anal.*, **14**:5 (2015), 2069–2094.

TRANSFORMATION OF THE EQUATION

The other way to avoid the spectrum sparseness condition is to transform (*some change of variables*) the parabolic equation in order **to decrease** the nonlinearity exponent θ. **The summetry property of the linear part of the equation must be preserved.** In this way J. Vukadinovic has constructed inertial manifolds [1, 2] for a Smoluchowski equation – a nonlinear Fokker–Planck equation on $S^m\ (m=1,2)$ and [3] for a class of diffusive Burgers equations on torus $[0,2\pi]^m\ (m=1,2)$. In paper [3] the Cole–Hopf transform has been employed.

But the last two methods are not being general. We can avoid the spectrum sparseness condition in some special cases only.

[1] J. Vukadinovic. *Nonlinearity*, **21** (2008), 1533–1545.

[2] J. Vukadinovic. *Comm. Math. Phys*., **285**:3 (2009), 975–990.

[3] J. Vukadinovic. *Discr. Cont. Dyn. Syst*., **29**:1 (2011), 327–341.

INERTIAL MANIFOLDS: NECESSITY

For a *fixed* admissible nonlinearity F, **there is only one known necessary condition** [1, 2] for the existence of an inertial manifold $M \subset H$ for the equation $u_t = -Au + F(u)$. For $u \in H$, we introduce the following notation:

1. $F'(u)$ is the Fréchet or Gâteaux derivative of the function F.
2. $\sigma(S_u)$ is the spectrum of the linear operator $S_u = F'(u) - A$ with compact resolvent.
3. E is the set of stationary points $u: -Au + F(u) = 0$.
4. $l(u) < \infty$ is the number (counting algebraic multiplicity) of eigenvalues $\lambda > 0$ in $\sigma(S_u)$ for $u \in E$.
5. $E_- = \{u \in E : \sigma(S_u) \bigcap (-\infty, 0] = \phi\}$.

NECESSITY LEMMA [1]. *If the equation* $u_t = -Au + F(u)$ *admits an inertial manifold* $M \subset H$, *then the number* $l(u_1) - l(u_2)$ *is even for any two points* $u_1, u_2 \in E_-$.

Sketch of proof. Let $Y = T_u M$ be the tangent space to M at a point $u \in E_-$, then $S_u Y \subset Y$. As $M \supset$attractor, then $\sigma(S_u|_Y)$ contains exactly $l(u)$ real values. Since $\dim Y = \dim M$, it follows that the number $\dim M - l(u)$ is even for any $u \in E_-$.

[1] A.V. Romanov. *Math. Notes*, **68**:3–4 (2000), 378–385.

[2] A. Eden, V. Kalantarov, and S. Zelik. *Russian Math. Surveys*, **68**:2 (2013), 199–226.

EQUATION WITH NONLOCAL DIFFUSION WITHOUT AN INERTIAL MANIFOLD

Consider the integro-differential parabolic equation

$$u_t = ((I + B)u_x)_x + f(x,u,u_x) \qquad (*)$$

on the unit circle Γ. Here $X = L^2(\Gamma)$, $I = \mathrm{id}$, $x \in \Gamma$, $f : \Gamma \times \mathbb{R}^2 \to \mathbb{R}$, and

$$(Bh)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \ln\left|\sin\frac{x+y}{2}\right| h(y)\,dy \quad (h \in X).$$

The self-adjoint operator $I + B \geq 0$ plays the role of a nonlocal degenerate diffusion coefficient, and $\partial_x B = J$, where $(Jh)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cot\frac{x+y}{2} h(y)\,dy$ is a slightly modified Hilbert integral operator.

THEOREM [1]. *For an appropriate choice of the function* $f \in C^\infty$, *Eq.* $(*)$ *generates a smooth dissipative semiflow in* $H = D(A^\theta)$, $3/4 < \theta < 1$, *and does not admit an inertial manifold* $M \subset H$.

[1] A.V. Romanov. *Math. Notes*, **96**:4 (2014), 548–555.

EQUATION WITH NONLOCAL DIFFUSION WITHOUT AN INERTIAL MANIFOLD – 1

In the course of proof, we use the properties of a Hilbert integral operator [1] and the perturbation theory technique [2] to construct a function $f(x,s,p)$ such that Eq. $(*)$ has stationary solutions $u_1 = 0, u_2 = 1, \quad u_1, u_2 \in E_-$, with $l(u_1) = 0, l(u_2) = 1.$ Then we apply the **necessity lemma.**

This example of a parabolic equation without an inertial manifold is much more realistic than the earlier-known examples but still is not completely natural.

[1] P.P. Zabreiko, et al. *Integral Equations. A Reference Text.* 1975.

[2] T. Kato. *Perturbation Theory for Linear Operators*, 1966.

NORMALLY HYPERBOLIC INERTIAL MANIFOLDS

For inertial manifolds with additional **normal hyperbolicity properties** [1], nonexistence examples can be constructed in the class of **reaction–diffusion systems**.

DEFINITION. An inertial manifold $M \subset H$ of the equation $u_t = -Au + F(u), H = D(A^\theta)$, is said to be *normally hyperbolic* if, for some (invariant with respect to the derivative Φ_t' of semiflow $\Phi_t : H \to H$) vector bundle $T_M H = TM \oplus N$, where TM is the tangent bundle of M, one has (for $t \geq 0$) the estimates

$$\|\Phi_t'(u)\xi\|_H \geq C^{-1} e^{-\gamma_1 t} \|\xi\|_H \quad (\xi \in T_u M),$$

$$\|\Phi_t'(u)\xi\|_H \leq C e^{-\gamma_2 t} \|\xi\|_H \quad (\xi \in N_u) \qquad (*)$$

for $u \in M$ with constants $C \geq 1$ and $0 < \gamma_1 < \gamma_2$ depending on M and u.

THE SENSE: the linearized semiflow contracts N more sharply then TM.

It is well known [1, 2] that normally hyperbolic invariant manifolds of dynamical systems are **structurally stable**.

[1] M. Hirsch, G. Pugh, and M. Shub. *Invariant manifolds*, Lecture Notes in Math., **583**, 1977.

[2] V.A. Pliss and G.R. Sell. *J. Diff. Equat.*, **169**, (2001) 396–492.

NORMALLY HYPERBOLIC INERTIAL MANIFOLDS: SUFFICIENCY

If C,γ_1,γ_2 in relations $(*)$ are independent of u, then we say that the manifold M *is absolutely normally hyperbolic*. If relations $(*)$ hold for $u\in E$, then we say that M is hyperbolic *at the stationary points*.

THEOREM [1]. *The spectral sparseness condition*

$$\sup_{n\geq 1}\frac{\mu_{n+1}-\mu_n}{\mu_{n+1}^\theta+\mu_n^\theta}=\infty,\ \ \textit{where}\ \{0<\mu_1<\mu_2<\ldots\}=\sigma(A),$$

for the semilinear parabolic equation

$$u_t=-Au+F(u) \qquad (**)$$

with the nonlinearity exponent $\theta\in[0,1)$ *in Hilbert space* X *implies the existence of an absolutely normally hyperbolic inertial manifold* M *in the phase space* $H=D(A^\theta)$.

THEOREM [2]. *The scalar reaction–diffusion equation* $\partial_t u=\Delta u+f(x,u)$, $f\in C^3$, *in cube* $\Omega=(0,2\pi)^3$ *and in rectangle* $\Omega=(0,a)\times(0,b)$ *with boundary conditions* (N),(D) *or* (P) *has normally hyperbolic at the stationary points inertial manifold* $M\subset L^2(\Omega)$.

RECENTLY [3]: the spatial averaging principle (abstract scheme) for Eq. (**) implies the existence of an normally hyperbolic inertial manifold in the phase space.

[1] R. Rosa, R. Temam. *ACTA Applicandae Mathematicae*, **45** (1996), 1–50.

[2] J. Mallet-Paret, G.R. Sell, and Z. Shao. *Indiana Univ. Math. J.*, **42**:3 (1993), 1027–1055.

[3] A. Kostianko and S. Zelik. *Comm. Pure Appl. Anal.*, **14**:5 (2015), 2069–2094.

NORMALLY HYPERBOLIC INERTIAL MANIFOLDS: NECESSITY

Let $M \subset H$ be an inertial manifold of the equation $u_t = -Au + F(u)$, let $\gamma \in \mathbb{R}$, and let $H(u,\gamma)$ be the finite-dimensional invariant subspace of the operator $S_u = F'(u) - A$ corresponding to the part of the spectrum $\sigma(S_u)$ with $\operatorname{Re}\lambda \geq \gamma$.

NECESSITY LEMMA [1]. *If M is normally hyperbolic on E, then*

$$\forall u \in E \ \exists \gamma = \gamma(u) < 0 : \dim H(u,\gamma) = \dim M .$$

Here $\gamma(u) = -(\gamma_1(u) + \gamma_2(u))/2$, the invariant subspaces $T_u M$ and N_u correspond to the parts of $\sigma(S_u)$ with $\operatorname{Re}\lambda \geq -\gamma_1(u)$ and $\operatorname{Re}\lambda \leq -\gamma_2(u)$, respectively, and $0 < \gamma_1(u) < \gamma_2(u)$ in the definition of normal hyperbolicity. If M is *absolutely normally hyperbolic*, then the constants $\gamma, \gamma_1, \gamma_2$ are independent of $u \in M$.

THEOREM [1]. *There exists a real-analytic function f such that the reaction–diffusion equation*

$$\partial_t u = \Delta u + f(x,u), \quad \Omega = (0,\pi)^4, \ \partial_n u\big|_{\partial\Omega} = 0$$

dissipative in $H = L^2(\Omega)$, does not admit a normally hyperbolic inertial manifold $M \subset H$.

The proof is based on the **necessity lemma** and uses the large multiplicity of the spectrum $\sigma(-\Delta)$ in $(0,\pi)^4$. The corresponding function $f : (0,\pi)^4 \times \mathbb{R} \to \mathbb{R}$ (polynomial in u) **is not constructed explicitly**.

[1] J. Mallet-Paret, G.R. Sell, and Z. Shao. *Indiana Univ. Math. J.*, **42**:3 (1993), 1027–1055.

PROBLEM:

Find 3D reaction–diffusion equations *with polynomial nonlinearity of degree* $\leq$ 3 that do not admit a normally hyperbolic inertial manifold

The restrictions on the dimension of the problem and the form of the nonlinearity are typical of the equations of chemical kinetics.

Consider the system of equations

$$\partial_t u_1 = \Delta u_1 + f_1(u_1, u_2), \quad \partial_t u_2 = \Delta u_2 + f_2(u_1, u_2), \qquad (*)$$

dissipative in $H = (L^2(\Omega))^2$, $\Omega = (0, \pi)^3$, with the condition $\partial_n u\big|_{\partial\Omega} = 0$ and with a smooth function $f: \mathbb{R}^2 \to \mathbb{R}^2$. For a stationary point $p \in \mathbb{R}^2$ of the vector field f, we set $\delta(p) = \left|\mathrm{Re}(\kappa_1 - \kappa_2)\right|$, where κ_1, κ_2 are the eigenvalues of the Jacobian matrix $f'(p)$. Note that $\delta(p) = 0$ for multiple or complex κ. The scalar operator $-\Delta$ has (for $x = (x_1, x_2, x_3) \in \Omega$) the eigenfunctions $\varphi(x) = \prod_{k=1}^{3} \cos l_k x_k$ $(l_k \in \mathbb{N} \cup 0)$ and the eigenvalues $\mu_n = l_1^2 + l_2^2 + l_3^2 \ \forall\, l_j \in \mathbb{Z}\}$ with multiplicity ν_n. It is well known [2] that $1 \le \mu_{n+1} - \mu_n \le 3$. One has the orthogonal decomposition

$$H = \sum_{n=0}^{\infty} \oplus H_n, \quad \Delta H_n = -\mu_n H_n, \quad \dim H_n = 2\nu_n.$$

OBSTRUCTION LEMMA [1]. *Assume that the vector field* $f = (f_1, f_2)$ *has four stationary points* $p_j \in \mathbb{R}^2 (j = 0,1,2,3)$ *with* $\delta(p_j) = j$. *Then system* $(*)$ *does not have a normally hyperbolic inertial manifold* $M \subset H$.

[1] A.V. Romanov. *Math. Notes*, **68**:3–4 (2000), 378–385.

[2] G.H. Hardy, E.M. Wright. *An introduction to the theory of numbers*, 1979.

SKETCH OF PROOF

We have $$\sigma(f'(p_0)) = s_0 \pm i\omega \ \ (s_0 \in \mathbb{R}, \omega \geq 0),$$

$$\sigma(f'(p_j)) = s_j \pm j/2 \ \ (1 \leq j \leq 3, \ s_j \in \mathbb{R}).$$

Subspaces H_n are invariant for the operators $S_j = \Delta + f'(p_j), 0 \leq j \leq 3.$ The spectrum $\sigma(S_j)$ is the union over of $n \geq 0$ the spectra of all $(2\times 2)-$matrices $-\mu_n I + f'(p_j)$ $(I = \mathrm{id})$:

$$\sigma(S_0) = \{-\mu_n + s_0 \pm i\omega\}_{n\geq 0}, \quad \sigma(S_1) = \{-\mu_n + s_1 \pm 1/2\}_{n\geq 0},$$

$$\sigma(S_2) = \{-\mu_n + s_2 \pm 1\}_{n\geq 0}, \quad \sigma(S_3) = \{-\mu_n + s_3 \pm 3/2\}_{n\geq 0}.$$

Let $$\chi(0,m) = \mathrm{card}\{\lambda \in \sigma(S_0)\text{: } \mathrm{Re}\,\lambda \geq -\mu_m + s_0\},$$

$$\chi(j,n) = \mathrm{card}\{\lambda \in \sigma(S_j)\text{: } \lambda \geq -\mu_n + s_j - j/2\},$$

$$\psi(j,n) = \mathrm{card}\{\lambda \in \sigma(S_j)\text{: } \lambda \geq -\mu_n + s_j + j/2\},$$

where $m, n \geq 0,\ 1 \leq j \leq 3,$ and **points of the spectrum are counted with multiplicities.**

Accordingly **necessity lemma** it is sufficiently to refute the conjecture:

$$\exists\, m, n_j \geq 0 \ (1 \leq j \leq 3) \ \ \forall j: \chi(0,m) = \chi(j,n_j) \text{ or } \chi(0,m) = \psi(j,n_j).$$

It may be done by arithmetical analysis of many variants.

EXAMPLE OF NONEXISTENCE OF A NORMALLY HYPERBOLIC INERTIAL MANIFOLD

Consider the system

$$\partial_t u_1 = \Delta u_1 + f_1(u_1, u_2), \quad \partial_t u_2 = \Delta u_2 + f_2(u_1, u_2)$$

in cube $\Omega = (0, \pi)^3$, under the condition $\partial_n u\big|_{\partial\Omega} = 0$ with the polynomial vector field

$$f_1(v_1, v_2) = kv_1(1 - av_1^2 + v_2^2), \quad f_2(v_1, v_2) = kv_2(1 - bv_2^2 - v_1^2),$$

where $a > 1$, $k, b > 0$, are constants. Dissipativity in $H = (L^2(\Omega))^2$ ("*vector sign condition*"): squares $|v_1| < R, |v_2| < R$ are positively invariant for ODE $v_t = f(v)$ when $R \geq R_0 > 0$.

PROPOSITION [1]. *There exist* k, a, b *such that this system does not have a normally hyperbolic inertial manifold* $M \subset H$.

[1] A.V. Romanov (unpublished).

SKETCH OF PROOF

Let us single out four stationary points

$$p_0 = (0,0), \quad p_1 = (a^{-1/2}, 0), \quad p_2 = \left(\left(\frac{b+1}{ab+1} \right)^{1/2}, \left(\frac{a-1}{ab+1} \right)^{1/2} \right), \quad p_3 = (0, b^{-1/2})$$

of the vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$. We have

$$f'(v) = k \begin{pmatrix} 1 - 3av_1^2 + v_2^2 & 2v_1 v_2 \\ -2v_1 v_2 & 1 - v_1^2 - 3bv_2^2 \end{pmatrix}$$

for $v \in \mathbb{R}^2$, and $\delta_0 = 0$, $\delta_1 = k(3 - a^{-1})$, $\delta_3 = k(3 + b^{-1})$,

$$\delta_2^2 = \frac{4k^2 a^2}{(ab+1)^2} - 16k^2 \frac{(a-1)(b+1)}{(ab+1)^2},$$

if the right part > 0 (otherwise $\delta_2 = 0$). We have $\delta_3 / \delta_1 = 3$, if $b = a/(6a-3)$. Let $g(a) = \delta_2(a)/\delta_1(a)$, then $g \in C[1, \infty)$, $g(1) = 3/4$, $g(\infty) = 4$ and so $g(a) = 2$ for some $a > 1$. Let now $k = a/(3a-1)$, then we see that $\delta(p_j) = j$ $(0 \le j \le 3)$.

The proposition follows now from the **obstruction lemma.**

PROBLEM:

Find 3D reaction–diffusion equations with an inertial manifold that is not normally hyperbolic

INERTIAL MANIFOLD THAT IS NOT NORMALLY HYPERBOLIC

Consider the system

$$\partial_t u_1 = \Delta u_1 + f_1(u_1, u_2), \quad \partial_t u_2 = \Delta u_2 + f_2(u_1, u_2), \qquad (*)$$

dissipative in $H = (L^2(\Omega))^2$, $\Omega = (0, \pi)^3$, with the boundary condition $\partial_n u\big|_{\partial\Omega} = 0$ and with the polynomial vector field

$$f_1(v_1, v_2) = v_1(a - v_1)(v_1 - b), \quad f_2(v_1, v_2) = v_2(c - v_2)(v_2 - d),$$

where a, b, c, d are constants. **THIS IS AN UNCOUPLED SYSTEM!**

PROPOSITION [1]. *For* $a = 2, b = \sqrt{3}, c = \sqrt{6}, d = \sqrt{2}$, *system* $(*)$ *has an inertial manifold* $M \subset H$ *but does not have a normally hyperbolic inertial manifold in* H.

Thus, we have presented an inertial manifold of system $(*)$ without the normal hyperbolicity property.

[1] A.V. Romanov (unpublished).

PROOF

By [1], each of the two equations in the system has an inertial manifold $M_j \subset L^2(\Omega)$ with Cartesian structure, and hence $M = M_1 \times M_2$ is an inertial manifold of the system in H. At the stationary points

$$p_0 = (0,0), \quad p_1 = (b,d), \quad p_2 = (a,c), \quad p_3 = (b,c)$$

of the vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$, we have

$$f'(p_0) = \begin{pmatrix} -ab & 0 \\ 0 & -cd \end{pmatrix} = \begin{pmatrix} -2\sqrt{3} & 0 \\ 0 & -2\sqrt{3} \end{pmatrix},$$

$$f'(p_1) = \begin{pmatrix} b(a-b) & 0 \\ 0 & d(c-d) \end{pmatrix} = \begin{pmatrix} 2\sqrt{3}-3 & 0 \\ 0 & 2\sqrt{3}-2 \end{pmatrix},$$

$$f'(p_2) = \begin{pmatrix} a(b-a) & 0 \\ 0 & c(d-c) \end{pmatrix} = \begin{pmatrix} 2\sqrt{3}-4 & 0 \\ 0 & 2\sqrt{3}-6 \end{pmatrix},$$

$$f'(p_2) = \begin{pmatrix} b(a-b) & 0 \\ 0 & c(d-c) \end{pmatrix} = \begin{pmatrix} 2\sqrt{3}-3 & 0 \\ 0 & 2\sqrt{3}-6 \end{pmatrix}.$$

We see that $\delta(p_j) = j \ (0 \le j \le 3)$, and the desired assertion follows from the **obstruction lemma.**

[1] J. Mallet-Paret and G.R. Sell. *J. Amer. Math. Soc.*, **1**:4 (1988), 805–866.

IS THE SPECTRUM SPARSENESS CONDITION

EQUIVALENT TO THE ABSOLUTELY NORMALLY HYPERBOLIC INERTIAL

MANIFOLD EXISTING ?

ABSOLUTELY NORMALLY HYPERBOLIC INERTIAL MANIFOLDS

Let us discuss the existence of such manifolds for the semilinear parabolic equation

$$u_t = -Au + F(u) \qquad (*)$$

in Hilbert space X with the phase space $H = D(A^{\theta}), 0 \le \theta < 1.$

PROBLEM. *Find a relationship between the following properties*:

(A) *The spectrum sparseness condition for the linear part Eq.* $(*)$:

$$\sup_{n\ge1} \frac{\mu_{n+1} - \mu_n}{\mu_{n+1}^{\theta} + \mu_n^{\theta}} = \infty, \textit{ where } \{0 < \mu_1 < \mu_2 < \ldots\} = \sigma(A);$$

(B) *For any admissible nonlinearity* F, *Eq.* $(*)$ *has an absolutely normally hyperbolic inertial manifold* $M \subset H$.

(C) *For any admissible nonlinearity* F, *Eq.* $(*)$ *has an inertial manifold* $M \subset H$ *absolutely normally hyperbolic at the stationary points.*

(D) *For any admissible nonlinearity* F, *Eq.* $(*)$ *has an inertial manifold* $M \subset H$.

PROPOSITION. *Properties* (*A*), (*B*), (*C*) *and* (*D*) *are equivalent.*

The implication (A)$\Rightarrow$(B) is known [1] and the implications (B)$\Rightarrow$(C), (C)$\Rightarrow$(D) are trivial. The implication (D)$\Rightarrow$(A) has been obtained in [2].

[1] R. Rosa, R. Temam. *ACTA Applicandae Mathematicae*, **45** (1996), 1–50.

[2] A. Eden, V. Kalantarov, and S. Zelik. *Russian Math. Surveys*, **68**:2 (2013), 99–226.

THE PARTICULAR CASE

One has slightly other picture for special classes of semilinear parabolic equations. Let us consider the scalar reaction–diffusion equation

$$\partial_t u = \Delta u + \eta f(u), \quad \eta > 0, \qquad (*)$$

in a bounded domain $\Omega \subset \mathbb{R}^m$ $(m \le 3)$ with the condition $\partial_n u\big|_{\partial\Omega} = 0$ and with a smooth function f. We assume that Eq. $(*)$ is dissipative in $H = L^2(\Omega)$. Let

$$\{0 \le \mu_1 < \mu_2 < \ldots\} = \sigma(-\Delta).$$

PROPOSITION [1]. *Let* $\mu_{n+1} - \mu_n \le K, n \ge 0,$ *and* $f'(p_0) - f'(p_1) = a > 0$ *for some* $p_0, p_1 \in \mathbb{R}$, $f(p_0) = f(p_1) = 0$. *Then Eq.* $(*)$ *with* $\eta > K/a$ *have no inertial manifold* $M \subset H$ *absolutely normally hyperbolic at the stationary points*.

For Eq. $(*)$ **we can affirm the equivalence of properties (A), (B), (C) only.**

[1] A.V. Romanov. *Math. Notes*, **68**:3–4 (2000), 378–385.

POSSIBLE GOALS

1. **Construct an example of a reaction–diffusion system without an inertial manifold.**

2. **Comprehensively study a relationship between the spectrum sparseness condition and absolutely normally hyperbolic inertial manifold existing for semilinear parabolic equations.**

3. **Successfully advancement of the spatial averaging principle (abstract scheme).**

4. **Comprehensively study the topic "inertial manifolds" for 2D Navier–Stokes equations.**

5. **The study of the topic "inertial manifolds" for reaction–diffusion equations on close manifolds.**

THANKS FOR ATTENTION