Formal solution of $\hbar-$ KP hierachy

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Tau-function

Hereafter we work with the $\hbar-$ KP hierarchy from K. Takasaki and T. Takebe, *Integrable hierarchies and dispersionless limit*, Rev. Math.Phys. **7** (1995) 743-808. Tau-function of the hierarchy is a function $\tau = \tau(\mathbf{t})$ depended on the infinite set of time variables $\mathbf{t} = \{t_1, t_2, \dots\}$. There is also contains $\hbar$ as a parameter but we will not write it explicitly. Below we use the notation

$$\tau^{[z_1,\dots,z_m]}(\mathbf{t}) = e^{\hbar(D(z_1)+\dots+D(z_m))}\,\tau,$$

where

$$D(z) = \sum_{k\geq 1} \frac{z^{-k}}{k}\,\partial_k; \quad \partial_k = \partial/\partial t_k. \tag{1}$$

Hirota relation

Let $\hbar > 0$. A function τ is called τ-function of $\hbar-$ KP hierarchy if it satisfies $\hbar-$Hirota functional relation.

$$(z_1 - z_2)\tau^{[z_1,z_2]}\tau^{[z_3]} + (z_2 - z_3)\tau^{[z_2,z_3]}\tau^{[z_1]} + (z_3 - z_1)\tau^{[z_3,z_1]}\tau^{[z_2]} = 0. \quad (2)$$

For $\hbar = 1$ it gives the ordinary Hirota relation.

Theorem

1. A function τ is τ-function of $\hbar-$ KP hierarchy if and only if

$$\hbar\partial_1 \log \frac{\tau^{[z_1]}}{\tau^{[z_2]}} = (z_2 - z_1)\left(\frac{\tau^{[z_1,z_2]}\tau}{\tau^{[z_1]}\tau^{[z_2]}} - 1\right). \quad (3)$$

2. A function τ is τ-function of $\hbar-$ KP hierarchy if and only if

$$\prod_{1\le i<j\le m}(z_j - z_i)\cdot \tau^{[z_1,\dots,z_m]}\tau^{m-1} = \det_{1\le j,k\le m}\left((z_j - \hbar\partial_1)^{k-1}\tau^{[z_j]}\right) \quad (4)$$

for any $m \ge 2$ and any $z_1, \dots, z_m$.

Schur polynomials

For description of τ-function we use Schur polynomials. An elementary Schur polynomial $h_k(\mathbf{t})$ depends from natural number k and infinite number of variable $\mathbf{t} = (t_1, t_2, \dots)$. It is defined by the generating series

$$\exp(\sum_{k\geq 1} t_k z^k) = \sum_{k\geq 0} h_k(\mathbf{t}) z^k.$$

An general Schur polynomial $s_\lambda(t)$ depends from a Young diagram $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_\ell]$ of degree $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ with a $\ell = \ell(\lambda) \geq 0$ rows of positive lengths $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$.

The general Schur polynomial is the determinant

$$s_\lambda(\mathbf{t}) = \det_{i,j=1,\dots,\ell(\lambda)} h_{\lambda_i - i + j}(\mathbf{t}).$$

Formal tau-function

Let us introduce the differential operator

$$\partial_k^{\hbar} = \sum_{l=1}^{k} \frac{\hbar^{l-1} k}{l!} \sum_{\substack{k_1,\dots,k_l \geq 1 \\ k_1+\dots+k_l=k}} \frac{\partial_{k_1} \dots \partial_{k_l}}{k_1 \dots k_l} = \partial_k + \hbar \sum_{l=1}^{k-1} \frac{k \partial_l \partial_{k-l}}{2l(k-l)} + O(\hbar^2). \quad (5)$$

In the KP theory the first variable t_1 is distinguished. Thus we will consider τ-function as evolution from a function $\tau(x,0)$ of one variable by $\hbar$- KP flows

$$\tau(x,0) \to \tau(x,\mathbf{t}) = f(x)\hat{\tau}(x+t_1, t_2, t_3, \dots).$$

A formal τ-function of $\hbar-$ KP hierarchy we mean a formal series for a function $\tau(x,\mathbf{t})$ of this type, that satisfy the Hirota equation by $\mathbf{t}$ for any x.

Our first goal is to find one-to-one correspondance between the set of all systems of differentiable functions $c_k(x), k = 0,1,2,\dots$ and the set of all formal τ-functions of $\hbar-$ KP hierarchy.

Formal solutions τ-function of $\hbar-$ KP hierarchy

Theorem

Let $\hbar \neq 0$ and $c_k(x)$, $k = 0, 1, 2, \ldots$, be arbitrary infinitely differentiable functions of x (with $c_0(x)$ being not identically 0). Put $c_\emptyset(x) = c_0(x)$ and

$$c_\lambda(x) = (c_0(x))^{1-\ell(\lambda)} \det_{1\le i,j\le \ell(\lambda)} \left[\sum_{k=0}^{j-1} (-\hbar)^k C_{j-1}^k \partial_x^k c_{\lambda_i - i + j - k}(x) \right] \tag{6}$$

for Young diagram $\lambda \neq \emptyset$. Then the series

$$\tau(x; \mathbf{t}) = \sum_\lambda c_\lambda(x) s_\lambda(\mathbf{t}/\hbar) \tag{7}$$

is a formal solution to the $\hbar$-KP hierarchy ($\hbar \neq 0$) where

$$\tau(x; \mathbf{0}) = c_0(x), \quad \partial_k^\hbar \tau(x; \mathbf{t}) \Big|_{\mathbf{t}=0} = \frac{k}{\hbar} c_k(x), \quad k \ge 1$$

and $c_1 = \partial_x c_0 - c_0 \partial_x \log f$.

Theorem

Let $\tau(x,\mathbf{t}) = f(x)\hat{\tau}(x+t_1, t_2, \dots)$ *be a tau-function of the* $\hbar$*-KP hierarchy with respect to the variables* t_j*, with* $\tau(x,\mathbf{0})$ *being an infinitely differentiable function of* x*. Then it has a representation by Young diagrams*

$$\tau(x;\mathbf{t}) = \sum_{\lambda} c_\lambda(x) s_\lambda(\mathbf{t}/\hbar), \tag{8}$$

where the coefficients are connected by the relations

$$c_\lambda(x) = (c_0(x))^{1-\ell(\lambda)} \det_{1\le i,j\le \ell(\lambda)} \left[\sum_{k=0}^{j-1} (-\hbar)^k C_{j-1}^k \partial_x^k c_{[\lambda_i - i + j - k]}(x) \right] \tag{9}$$

for $\lambda \neq \emptyset$*,* $c_\emptyset(x) = c_0(x)$ *and* $c_1 = \partial_x c_0 - c_0 \partial_x \log f$*.*

$\hbar-$ KP hierarchy

For many applications in physics and mathematics one needs to deal with logarithm of the tau-function rather than with the tau-function itself. Let us put

$$F(x;\mathbf{t}) = \hbar^2 \log \tau(x;\mathbf{t}). \tag{10}$$

Then the Hirota equations on $\tau(x;\mathbf{t})$ go to $\hbar-$ KP hierarchy on $F(x;\mathbf{t})$. That is

$$e^{\Delta(z_1)\Delta(z_2)F} = 1 - \frac{\Delta(z_1)\partial_x F - \Delta(z_2)\partial_x F}{z_1 - z_2}, \tag{11}$$

where

$$\Delta(z) = \frac{e^{\hbar D(z)} - 1}{\hbar}. \tag{12}$$

For $\hbar = 1$ this is the ordinary KP hierarchy.
For $\hbar = 0$ this is the dispersionless KP hierarchy.

Our goal is a formula, expressing any solution of $\hbar-$ KP hierarchy by *Cauchy-like data* $f_k(x) = \partial_k^\hbar F(x; t_1, t_2, \ldots)\Big|_{\mathbf{t}=0}$.

The $\hbar$-KP hierarchy in terms of $\partial_k^\hbar$

Define some combinatorial constants $\tilde{P}_{ij}(s_1,\dots,s_m)$ as the number of sequences of positive integers $(i_1,\dots,i_m)$ and $(j_1,\dots,j_m)$ such that $i_1+\dots+i_m=i$, $j_1+\dots+j_m=j$ and $s_k=i_k+j_k-1$. Put

$$P_{ij}(s_1,\,\dots,s_m)=\frac{(-1)^{m+1}ij}{m\,s_1\dots s_m}\,\tilde{P}_{ij}(s_1,\,\dots,s_m).$$

Theorem

The $\hbar$-KP hierarchy is equivalent to the system of equations

$$\partial_i^\hbar\partial_j^\hbar F=\sum_{m\geq 1}\sum_{\substack{s_1,\dots,s_m\geq 1\\ s_1+\dots+s_m=i+j-m}} P_{ij}(s_1,\,\dots,s_m)\,\partial\partial_{s_1}^\hbar F\,\dots\,\partial\partial_{s_m}^\hbar F, \tag{13}$$

for the function $F=F(x;\mathbf{t})$, where $\partial=\partial_1$.

Combinatorial constants

Let $K_l(l^1, \dots, l^r)$ be the number of partitions of a set of l elements into ordered groups of $l^1, \dots, l^r > 0$ elements.

Define the constants $P^{\hbar}_{i_1\dots i_k}\left(\begin{array}{c} s_1 \dots s_m \\ l_1 \dots l_m \end{array}\right)$ from integer positive $m, \{i_r\}, \{s_r\}, \{l_r\}$ by the following recurrence relations:

1) $P^{\hbar}_{i_1,i_2}\left(\begin{array}{c} s_1 \dots s_m \\ 1 \dots 1 \end{array}\right) = P_{i_1 i_2}(s_1, \dots, s_m)$ and $P^{\hbar}_{i_1,i_2}\left(\begin{array}{c} s_1 \dots s_m \\ l_1 \dots l_m \end{array}\right) = 0$, if $\prod_{j=1}^{m} l_j > 1$.

2) $P^{\hbar}_{i_1\dots i_r}\left(\begin{array}{c} x_1 \dots x_v \\ y_1 \dots y_v \end{array}\right) = \sum P^{\hbar}_{i_1\dots i_{r-1}}\left(\begin{array}{c} s_1 \dots s_m \\ l_1 \dots l_m \end{array}\right) \frac{\hbar^{\nu(k_1,\dots,k_m)-1} i_r}{[k_1\dots k_m]}$
$\times K_{l_1}(l_1^1, \dots, l_1^{n_1}) P_{s_1 k_1}(s_1^1 \dots s_{n_1}^1) \dots K_{l_m}(l_m^1, \dots, l_m^{n_m}) P_{s_m k_m}(s_1^m \dots s_{n_m}^m)$,
where $\nu(k_1, \dots, k_n)$ is the number of positive numbers between k_i and $[k_1, \dots, k_n)] = \prod_{i=1}^{n} \max\{k_i, 1\}$.

The summation is carried over all sets of integer numbers such that

$$(x_1 \dots x_v) = (s^1_1, \dots\ s^1_{n_1}, s^2_1, \dots, s^2_{n_2}, \dots, s^m_1, \dots, s^m_{n_m}), \quad s_i = \sum_{j=1}^{n_i} s^i_j,$$

$$(y_1 \dots y_v) = (l^1_1 + 1, \dots, l^1_{n_1} + 1, l^2_1 + 1, \dots l^2_{n_2} + 1, \dots, l^m_1 + 1, \dots, l^m_{n_m} + 1),$$

$$l_i = \sum_{j=1}^{n_i} l^i_j, \sum_{i=1}^{m}(s_i + l_i) = \sum_{j=1}^{r-1} i_j, \quad \sum_{i=1}^{m} k_i = i_r, \quad \sum_{i=1}^{n_j} s^j_i = k_j + s_j.$$

Theorem

The $\hbar$-KP hierarchy is equivalent to the system of equations for $r \geq 2$

$$\partial^\hbar_{i_1} \dots \partial^\hbar_{i_r} F = \sum_{m \geq 1} \sum_{\substack{s_1 + l_1 + \dots + s_m + l_m \\ = i_1 + \dots + i_r \\ 1 \leq s_i;\ 1 \leq l_i \leq r-1}} P^\hbar_{i_1 \dots i_r} \begin{pmatrix} s_1 \dots s_m \\ l_1 \dots l_m \end{pmatrix} \partial^{l_1} \partial^\hbar_{s_1} F \ \dots\ \partial^{l_m} \partial^\hbar_{s_m} F \tag{14}$$

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Variables $t^{\hbar}_{\lambda}$, corresponding to $\partial^{\hbar}_{\lambda}$

A Young diagram $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_\ell]$ is described by $\lambda = (1^{r_1} 2^{r_2} \dots n^{r_n} \dots)$, where $r_i = card\{j | \lambda_j = i\}$. Denote by $\rho(\lambda) = \lambda_1 \lambda_2 \dots \lambda_\ell$ and $\sigma(\lambda) = \prod\limits_{n \geq 1} r_n!$. We put also $\partial^{\hbar}_{\lambda} = \partial^{\hbar}_{\lambda_1} \partial^{\hbar}_{\lambda_2}, \dots, \partial^{\hbar}_{\lambda_\ell}$.

Let us consider the basis

$$m_\lambda(x_1, x_2, \dots, x_n) = \frac{1}{(n - \ell(\lambda))!\, \sigma(\lambda)} \sum_{s \in S_n} x_1^{\lambda_{s(1)}} x_2^{\lambda_{s(2)}} \dots x_n^{\lambda_{s(n)}}$$

in the space of symmetrical polynomials from $x_1, x_2, \dots$.

These polynomials are linear combinations from symmetrical polynomials $t_k = \frac{1}{k} \sum_i x_i^k$. This gives new polynomials $m_\lambda(\mathbf{t})$ from $\mathbf{t} = (t_1, t_2, \dots)$.

The first few functions m_λ are:

$$m_{(1)}(\mathbf{t}) = t_1,$$

$$m_{(2)}(\mathbf{t}) = 2t_2, \quad m_{(1^2)}(\mathbf{t}) = \frac{1}{2}\, t_1^2 - t_2,$$

$$m_{(3)}(\mathbf{t}) = 3t_3, \quad m_{(12)}(\mathbf{t}) = 2t_2 t_1 - 3t_3, \quad m_{(1^3)}(\mathbf{t}) = \frac{1}{6}\, t_1^3 - t_2 t_1 + t_3. \tag{15}$$

Let us put

$$t_\lambda^\hbar := \frac{\sigma(\lambda)}{\rho(\lambda)}\,\hbar^{\ell(\lambda)} m_\lambda(\mathbf{t}/\hbar). \quad (16)$$

The first few are (see (15)):

$$t_{(1)}^\hbar = t_1,$$

$$t_{(2)}^\hbar = t_2, \quad t_{(1^2)}^\hbar = t_1^2 - 2\hbar t_2, \quad (17)$$

$$t_{(3)}^\hbar = t_3, \quad t_{(21)}^\hbar = t_2 t_1 - \tfrac{3}{2}\hbar t_3, \quad t_{(1^3)}^\hbar = t_1^3 - 6\hbar t_2 t_1 + 6\hbar^2 t_3.$$

Theorem

Any formal series $F(\mathbf{t}) = F(t_1, t_2, \dots)$ has a representation in form of formal series

$$F(\mathbf{t}) = \sum_\lambda \partial_\lambda^\hbar F(\mathbf{t}')\Big|_{\mathbf{t}'=0} \frac{t_\lambda^\hbar}{\sigma(\lambda)}, \quad (18)$$

Construction of formal solutions

Consider now any family of infinitely differentiable functions

$$f^{\hbar}_{[k]}(x) = f^{\hbar}_{k}(x) \quad (k = 1, 2, \dots).$$

For other Young diagrams λ we put

$$f^{\hbar}_{\lambda}(x) = \sum_{m \geq 1} \sum_{\substack{s_1 + l_1 + \dots + s_m + l_m = |\lambda| \\ 1 \leq s_i;\ 1 \leq l_i \leq \ell(\lambda) - 1}} P^{\hbar}_{\lambda} \begin{pmatrix} s_1 \dots s_m \\ l_1 \dots l_m \end{pmatrix} \partial^{l_1} f_{s_1}(x) \dots \partial^{l_m} f_{s_m}(x), \tag{19}$$

where

$$P^{\hbar}_{\lambda} \begin{pmatrix} s_1 \dots s_m \\ l_1 \dots l_m \end{pmatrix} = P^{\hbar}_{\lambda_1 \dots \lambda_r} \begin{pmatrix} s_1 \dots s_m \\ l_1 \dots l_m \end{pmatrix}$$

for $\lambda = [\lambda_1, \dots, \lambda_\ell]$.

Theorem

For any $\hbar$ *and any family of smooth or formal functions*

$$\boldsymbol{f} = \{f_0(x), f_1(x), f_2(x), \ldots\}$$

there exists a unique solution $F(x;\mathbf{t})$ *of the* $\hbar$-KP *hierarchy such that*

$$F(x;\mathbf{0}) = f_0(x) \quad \text{and} \quad \partial_k^{\hbar} F(x; t_1, t_2, \ldots)\Big|_{\mathbf{t}=0} = f_k(x).$$

This solution has the form

$$F(x;\mathbf{t}) = f_0(x) + \sum_{|\lambda|\geq 1} \frac{f_\lambda^{\hbar}(x)}{\sigma(\lambda)}\, t_\lambda^{\hbar}. \tag{20}$$